A general type of twisted anomaly cancellation formulas

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Abstract

For even dimensional manifolds, we prove some twisted anomaly cancellation formulas which generalize some well-known cancellation formulas. For odd dimensional manifolds, we obtain some modularly invariant characteristic forms by the Chern-Simons transgression and we also get some twisted anomaly cancellation formulas.

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1 Introduction

In 1983, the physicists Alvarez-Gaumé and Witten [AW] discovered the "miraculous cancellation" formula for gravitational anomaly which reveals a beautiful relation between the top components of the Hirzebruch $\widehat{L}$-form and $\widehat{A}$-form of a 12-dimensional smooth Riemannian manifold. Kefeng Liu [Li] established higher dimensional "miraculous cancellation" formulas for $(8k+4)$-dimensional Riemannian manifolds by developing modular invariance properties of characteristic forms. These formulas could be used to deduce some divisibility results. In [HZ1], [HZ2], for each $(8k+4)$-dimensional smooth Riemannian manifold, a more general cancellation formula that involves a complex line bundle was established. This formula was applied to spin^c manifolds, then an analytic Ochanine congruence formula was derived. For $(8k+2)$ and $(8k+6)$-dimensional smooth Riemannian manifolds, F. Han and X. Huang [HH] obtained some cancellation formulas. They also got a general type of cancellation formulas.

On the other hand, motivated by the Chern-Simons theory, in [CH], Qingtao Chen and Fei Han computed the transgressed forms of some modularly invariant characteristic forms, which are related to the elliptic genera. They studied the modularity properties of these secondary characteristic forms and relations among them. They also got an anomaly cancellation formula for 11-dimensional manifolds. In [W], the author computed the transgressed forms of some modularly invariant characteristic forms, which are related to the "twisted" elliptic genera and studied the modularity properties of these secondary characteristic forms and relations among them. We also got some twisted anomaly cancellation formulas on some odd dimensional manifolds. The purpose of paper is to prove more general cancellation formulas for even and

odd dimensional manifolds. We hope that these new general cancellation formulas obtained here could be applied somewhere.

This paper is organized as follows: In Section 2, we review some knowledge on characteristic forms and modular forms that we are going to use. In Section 3, we prove some general cancellation formulas which involve two complex line bundles and generalize some well-known cancellation formulas for even dimensional manifolds. In Section 4, we apply the Chern-Simons transgression to characteristic forms with modularity properties which are related to the "twisted" elliptic genera and obtain some interesting secondary characteristic forms with modularity properties. We also get two twisted cancellation formulas for 9- and 11-dimensional manifolds.

2 characteristic forms and modular forms

The purpose of this section is to review the necessary knowledge on characteristic forms and modular forms that we are going to use.

2.1 characteristic forms. Let M be a Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM}. Let $\widehat{A}(TM, \nabla^{TM})$ and $\widehat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined respectively by (cf. [Z])

$$\widehat{A}(TM, \nabla^{TM}) = \det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh(\frac{\sqrt{-1}}{4\pi}R^{TM})}\right),$$

$$\widehat{L}(TM, \nabla^{TM}) = \det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh(\frac{\sqrt{-1}}{4\pi}R^{TM})}\right). \tag{2.1}$$

Let E, F be two Hermitian vector bundles over M carrying Hermitian connection ∇^E, ∇^F respectively. Let $R^E = (\nabla^E)^2$ (resp. $R^F = (\nabla^F)^2$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as

$$\mathrm{ch}(G, \nabla^G) = \mathrm{tr}\left[\exp(\frac{\sqrt{-1}}{2\pi}R^E)\right] - \mathrm{tr}\left[\exp(\frac{\sqrt{-1}}{2\pi}R^F)\right]. \tag{2.2}$$

For any complex number t, let

$$\wedge_t(E) = \mathbf{C}|_M + tE + t^2\wedge^2(E) + \cdots,\ S_t(E) = \mathbf{C}|_M + tE + t^2S^2(E) + \cdots$$

denote respectively the total exterior and symmetric powers of E, which live in $K(M)[[t]]$. The following relations between these operations hold,

$$S_t(E) = \frac{1}{\wedge_{-t}(E)},\ \ \wedge_t(E - F) = \frac{\wedge_t(E)}{\wedge_t(F)}. \tag{2.3}$$

Moreover, if $\{\omega_i\}, \{\omega'_j\}$ are formal Chern roots for Hermitian vector bundles E, F respectively, then

$$\mathrm{ch}(\wedge_t(E)) = \prod_i (1 + e^{\omega_i} t). \tag{2.4}$$

Then we have the following formulas for Chern character forms,

$$\mathrm{ch}(S_t(E)) = \frac{1}{\prod_i (1 - e^{\omega_i} t)}, \ \mathrm{ch}(\wedge_t(E - F)) = \frac{\prod_i (1 + e^{\omega_i} t)}{\prod_j (1 + e^{\omega'_j} t)}. \tag{2.5}$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W, then its complexification $W_{\mathbf{C}} = W \otimes \mathbf{C}$ is a complex vector bundle over M carrying a canonical induced Hermitian metric from that of W, as well as a Hermitian connection $\nabla^{W_{\mathbf{C}}}$ induced from ∇^W. If E is a vector bundle (complex or real) over M, set $\widetilde{E} = E - \mathrm{dim} E$ in $K(M)$ or $KO(M)$.

2.2 Some properties about the Jacobi theta functions and modular forms

We first recall the four Jacobi theta functions are defined as follows(cf. [Ch]):

$$\theta(v, \tau) = 2q^{\frac{1}{8}} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^j)(1 - e^{-2\pi\sqrt{-1}v} q^j)], \tag{2.6}$$

$$\theta_1(v, \tau) = 2q^{\frac{1}{8}} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^j)(1 + e^{-2\pi\sqrt{-1}v} q^j)], \tag{2.7}$$

$$\theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^{j-\frac{1}{2}})(1 - e^{-2\pi\sqrt{-1}v} q^{j-\frac{1}{2}})], \tag{2.8}$$

$$\theta_3(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^{j-\frac{1}{2}})(1 + e^{-2\pi\sqrt{-1}v} q^{j-\frac{1}{2}})], \tag{2.9}$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half complex plane. Let

$$\theta'(0, \tau) = \frac{\partial \theta(v, \tau)}{\partial v}|_{v=0}. \tag{2.10}$$

Then the following Jacobi identity (cf. [Ch]) holds,

$$\theta'(0, \tau) = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau). \tag{2.11}$$

Denote $SL_2(\mathbf{Z}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a, b, c, d \in \mathbf{Z}, \ ad - bc = 1 \right\}$ the modular group. Let $S = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$, $T = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ be the two generators of $SL_2(\mathbf{Z})$. They act on $\mathbf{H}$ by $S\tau = -\frac{1}{\tau}$, $T\tau = \tau + 1$. One has the following transformation laws of theta functions under the actions of S and T (cf. [Ch]):

$$\theta(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta(v, \tau), \quad \theta(v, -\frac{1}{\tau}) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau); \tag{2.12}$$

$$\theta_1(v,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta_1(v,\tau),\quad \theta_1(v,-\frac{1}{\tau})=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}\theta_2(\tau v,\tau); \tag{2.13}$$

$$\theta_2(v,\tau+1)=\theta_3(v,\tau),\quad \theta_2(v,-\frac{1}{\tau})=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}\theta_1(\tau v,\tau); \tag{2.14}$$

$$\theta_3(v,\tau+1)=\theta_2(v,\tau),\quad \theta_3(v,-\frac{1}{\tau})=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}\theta_3(\tau v,\tau). \tag{2.15}$$

Differentiating the above transformation formulas, we get that

$$\theta'(v,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta'(v,\tau),$$

$$\theta'(v,-\frac{1}{\tau})=\frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}(2\pi\sqrt{-1}\tau v\theta(\tau v,\tau)+\tau\theta'(\tau v,\tau));$$

$$\theta_1'(v,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta_1'(v,\tau),$$

$$\theta_1'(v,-\frac{1}{\tau})=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}(2\pi\sqrt{-1}\tau v\theta_2(\tau v,\tau)+\tau\theta_2'(\tau v,\tau));$$

$$\theta_2'(v,\tau+1)=\theta_3'(v,\tau),$$

$$\theta_2'(v,-\frac{1}{\tau})=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}(2\pi\sqrt{-1}\tau v\theta_1(\tau v,\tau)+\tau\theta_1'(\tau v,\tau));$$

$$\theta_3'(v,\tau+1)=\theta_2'(v,\tau),$$

$$\theta_3'(v,-\frac{1}{\tau})=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}(2\pi\sqrt{-1}\tau v\theta_3(\tau v,\tau)+\tau\theta_3'(\tau v,\tau)) \tag{2.16}$$

Therefore

$$\theta'(0,-\frac{1}{\tau})=\frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}\tau\theta'(0,\tau). \tag{2.17}$$

Definition 2.1 A modular form over Γ, a subgroup of $SL_2(\mathbf{Z})$, is a holomorphic function $f(\tau)$ on $\mathbf{H}$ such that

$$f(g\tau):=f\left(\frac{a\tau+b}{c\tau+d}\right)=\chi(g)(c\tau+d)^kf(\tau),\quad \forall g=\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in\Gamma, \tag{2.18}$$

where $\chi:\Gamma\to\mathbf{C}^\star$ is a character of Γ. k is called the weight of f.
Let

$$\Gamma_0(2)=\left\{\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in SL_2(\mathbf{Z})\mid c\equiv 0\ (\text{mod } 2)\right\},$$

$$\Gamma^0(2)=\left\{\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in SL_2(\mathbf{Z})\mid b\equiv 0\ (\text{mod } 2)\right\},$$

$$\Gamma_\theta=\left\{\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in SL_2(\mathbf{Z})\mid \left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\equiv\left(\begin{array}{cc} 1 & 0\\ 0 & 1\end{array}\right)\text{or}\left(\begin{array}{cc} 0 & 1\\ 1 & 0\end{array}\right)\ (\text{mod } 2)\right\}$$

be the three modular subgroups of $SL_2(\mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are $T,\ ST^2ST$, the generators of $\Gamma^0(2)$ are $STS,\ T^2STS$ and the generators of Γ_θ are $S,\ T^2$ (cf.[Ch]).

If Γ is a modular subgroup, let $\mathcal{M}_{\mathbf{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients. Writing $\theta_j = \theta_j(0,\tau),\ 1 \leq j \leq 3$, we introduce six explicit modular forms (cf. [Li]),

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4), \quad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4,$$

$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4), \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4,$$

$$\delta_3(\tau) = \frac{1}{8}(\theta_1^4 - \theta_2^4), \quad \varepsilon_3(\tau) = -\frac{1}{16}\theta_1^4\theta_2^4.$$

They have the following Fourier expansions in $q^{\frac{1}{2}}$:

$$\delta_1(\tau) = \frac{1}{4} + 6q + \cdots, \quad \varepsilon_1(\tau) = \frac{1}{16} - q + \cdots,$$

$$\delta_2(\tau) = -\frac{1}{8} - 3q^{\frac{1}{2}} + \cdots, \quad \varepsilon_2(\tau) = q^{\frac{1}{2}} + \cdots,$$

$$\delta_3(\tau) = -\frac{1}{8} + 3q^{\frac{1}{2}} + \cdots, \quad \varepsilon_3(\tau) = -q^{\frac{1}{2}} + \cdots,$$

where the " $\cdots$ " terms are the higher degree terms, all of which have integral coefficients. They also satisfy the transformation laws,

$$\delta_2(-\frac{1}{\tau}) = \tau^2\delta_1(\tau), \qquad \varepsilon_2(-\frac{1}{\tau}) = \tau^4\varepsilon_1(\tau), \tag{2.19}$$

$$\delta_2(\tau+1) = \delta_3(\tau), \qquad \varepsilon_2(\tau+1) = \varepsilon_3(\tau). \tag{2.20}$$

Lemma 2.2 ([Li]) *$\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, while $\delta_3(\tau)$ (resp. $\varepsilon_3(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_\theta(2)$ and moreover $\mathcal{M}_{\mathbf{R}}(\Gamma^0(2)) = \mathbf{R}[\delta_2(\tau), \varepsilon_2(\tau)]$.*

3 A general type of cancellation formulas for even dimensional manifolds

Let M be a $2d$ dimensional Riemannian manifold and $\xi^0,\ \ \xi$ be rank two real oriented Euclidean vector bundles over M carrying with Euclidean connections ∇^{ξ^0}, ∇^ξ. Set

$$\Theta_1(T_CM, m_0\xi_C^0, \xi_C) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_CM} - m_0\widetilde{\xi_C^0}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^m}(\widetilde{T_CM} - m_0\widetilde{\xi_C^0} - 2\widetilde{\xi_C})$$

$$\otimes \bigotimes_{r=1}^{\infty} \wedge_{q^{r-\frac{1}{2}}}(\widetilde{\xi_C}) \otimes \bigotimes_{s=1}^{\infty} \wedge_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_C}),$$

$$\Theta_2(T_CM, m_0\xi_C^0, \xi_C) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_CM} - m_0\widetilde{\xi_C^0}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{-q^{m-\frac{1}{2}}}(\widetilde{T_CM} - m_0\widetilde{\xi_C^0} - 2\widetilde{\xi_C})$$

$$\otimes \bigotimes_{r=1}^{\infty} \wedge_{q^{r-\frac{1}{2}}}(\widetilde{\xi_C}) \otimes \bigotimes_{s=1}^{\infty} \wedge_{q^s}(\widetilde{\xi_C}), \tag{3.1}$$

Clearly, $\Theta_1(T_CM, m_0\xi_C^0, \xi_C)$ and $\Theta_2(T_CM, m_0\xi_C^0, \xi_C)$ admit formal Fourier expansion in $q^{\frac{1}{2}}$ as

$$\Theta_1(T_CM, m_0\xi_C^0, \xi_C) = A_0(T_CM, m_0\xi_C^0, \xi_C) + A_1(T_CM, m_0\xi_C^0, \xi_C)q^{\frac{1}{2}} + \cdots,$$

$$\Theta_2(T_CM, m_0\xi_C^0, \xi_C) = B_0(T_CM, m_0\xi_C^0, \xi_C) + B_1(T_CM, m_0\xi_C^0, \xi_C)q^{\frac{1}{2}} + \cdots, \tag{3.2}$$

where the A_j and B_j are elements in the semi-group formally generated by Hermitian vector bundles over M. Moreover, they carry canonically induced Hermitian connections. Let $c_0 = e(\xi, \nabla^{\xi^0})$ and $c = e(\xi, \nabla^{\xi})$ be the Euler forms of ξ^0, ξ canonically associated to ∇^{ξ^0} ∇^{ξ} respectively. If ω is a differential form over M, we denote $\omega^{(2d)}$ its top degree component. Let n be a nonnegative integer and satisfy $d - \left(2n + \frac{1-(-1)^d}{2}\right) > 0$, then we have

Theorem 3.1 *The following identity holds,*

$$\left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\frac{c}{2}} \frac{(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}}{(\cosh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}} \right\}^{(2d)}$$

$$= 2^{\frac{3}{2}d-n-\frac{1-(-1)^d}{4}} \sum_{r=0}^{[\frac{m_1}{2}]} 2^{-6r} \left\{ \widehat{A}(TM, \nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}} \right.$$

$$\left. \cdot\mathrm{ch}(b_r(T_CM, (2n + \frac{1-(-1)^d}{2})\xi_C^0, \xi_C)) \right\}^{(2d)}, \tag{3.3}$$

where $m_1 = \frac{d}{2} - n - \frac{1-(-1)^d}{4}$ *and each* $b_r(T_CM, (2n + \frac{1-(-1)^d}{2})\xi_C^0, \xi_C)$, $0 \le r \le [\frac{m_1}{2}]$, *is a canonical integral linear combination of* $B_j(T_CM, (2n + \frac{1-(-1)^d}{2})\xi_C^0, \xi_C)$, $0 \le j \le r$.

Proof. Let $\{\pm 2\pi\sqrt{-1}x_j |\ 1 \le j \le d\}$ be the Chern roots of T_CM and $c_0 = 2\pi\sqrt{-1}u'$. $c = 2\pi\sqrt{-1}u$. Set

$$Q_1(\tau) = \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\frac{c}{2}} \frac{(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}}{(\cosh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}} \mathrm{ch}(\Theta_1(T_CM, (2n + \frac{1-(-1)^d}{2})\xi_C^0, \xi_C)), \tag{3.4}$$

$$Q_2(\tau)=\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}\mathrm{ch}(\Theta_2(T_CM,(2n+\frac{1-(-1)^d}{2})\xi_C^0,\xi_C)), \tag{3.5}$$

Let $\Theta_1(T_CM,\xi_C)=\Theta_1(T_CM,m_0C^2,\xi_C),\ \Theta_2(T_CM,\xi_C)=\Theta_2(T_CM,m_0C^2,\xi_C)$. Then

$$Q_1(\tau)=\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2\frac{c}{2}}\mathrm{ch}(\Theta_1(T_CM,,\xi_C))$$

$$\cdot\left[\frac{\cosh\frac{c_0}{2}}{\sinh\frac{c_0}{2}}\mathrm{ch}\left(\bigotimes_{n=1}^{\infty}S_{q^n}(\widetilde{\xi_C^0})\otimes\bigotimes_{m=1}^{\infty}\wedge_{q^m}(\widetilde{\xi_C^0})\right)\right]^{-2n-\frac{1-(-1)^d}{2}}. \tag{3.6}$$

By Proposition 2.5 in [HZ2], we have

$$\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2\frac{c}{2}}\mathrm{ch}(\Theta_1(T_CM,,\xi_C))$$

$$=2^d\left\{\prod_{j=1}^{d}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\frac{\theta_1(x_j,\tau)}{\theta_1(0,\tau)}\right)\frac{\theta_1^2(0,\tau)}{\theta_1^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\right\}. \tag{3.7}$$

Direct computations show that

$$\frac{\cosh\frac{c_0}{2}}{\sinh\frac{c_0}{2}}\mathrm{ch}\left(\bigotimes_{n=1}^{\infty}S_{q^n}(\widetilde{\xi_C^0})\otimes\bigotimes_{m=1}^{\infty}\wedge_{q^m}(\widetilde{\xi_C^0})\right)=\frac{1}{\pi\sqrt{-1}}\frac{\theta'(0,\tau)}{\theta(u',\tau)}\frac{\theta_1(u',\tau)}{\theta_1(0,\tau)}. \tag{3.8}$$

By (3.6)-(3.8), we have

$$Q_1(\tau)=2^d(\pi\sqrt{-1})^{2n+\frac{1-(-1)^d}{2}}\left\{\prod_{j=1}^{d}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\frac{\theta_1(x_j,\tau)}{\theta_1(0,\tau)}\right)\right.$$

$$\left.\left(\frac{\theta(u',\tau)}{\theta'(0,\tau)}\frac{\theta_1(0,\tau)}{\theta_1(u',\tau)}\right)^{2n+\frac{1-(-1)^d}{2}}\cdot\frac{\theta_1^2(0,\tau)}{\theta_1^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\right\}. \tag{3.9}$$

Similarly,

$$Q_2(\tau)=\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}\mathrm{ch}(\Theta_2(T_CM,\xi_C))$$

$$\cdot\left[(\sinh\frac{c_0}{2})\mathrm{ch}\left(\bigotimes_{n=1}^{\infty}S_{q^n}(-\widetilde{\xi_C^0})\otimes\bigotimes_{m=1}^{\infty}\wedge_{-q^{m-\frac{1}{2}}}(-\widetilde{\xi_C^0})\right)\right]^{2n+\frac{1-(-1)^d}{2}}; \tag{3.10}$$

$$\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}\mathrm{ch}(\Theta_2(T_CM,\xi_C))$$

$$=\left(\prod_{j=1}^{d}x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\frac{\theta_2(x_j,\tau)}{\theta_2(0,\tau)}\right)\frac{\theta_2^2(0,\tau)}{\theta_2^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}; \tag{3.11}$$

$$\sinh\frac{c_0}{2}\mathrm{ch}\left(\bigotimes_{n=1}^{\infty}S_{q^n}(-\widetilde{\xi_C^0})\otimes\bigotimes_{m=1}^{\infty}\wedge_{-q^{m-\frac{1}{2}}}(-\widetilde{\xi_C^0})\right)=\sqrt{-1}\pi\frac{\theta(u',\tau)}{\theta'(0,\tau)}\frac{\theta_2(0,\tau)}{\theta_2(u',\tau)}, \tag{3.12}$$

so we have

$$Q_2(\tau) = (\sqrt{-1}\pi)^{2n+\frac{1-(-1)^d}{2}} \left(\prod_{j=1}^{d} x_j \frac{\theta'(0,\tau)}{\theta(x_j,\tau)} \frac{\theta_2(x_j,\tau)}{\theta_2(0,\tau)}\right)$$

$$\cdot \left(\frac{\theta(u',\tau)}{\theta'(0,\tau)} \frac{\theta_2(0,\tau)}{\theta_2(u',\tau)}\right)^{2n+\frac{1-(-1)^d}{2}} \frac{\theta_2^2(0,\tau)}{\theta_2^2(u,\tau)} \frac{\theta_3(u,\tau)}{\theta_3(0,\tau)} \frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}. \tag{3.13}$$

Let $P_1(\tau) = Q_1(\tau)^{(2d)}$, $P_2(\tau) = Q_2(\tau)^{(2d)}$. By (2.12)-(2.15) and (2.17), then $P_1(\tau)$ is a modular form of weight $d - (2n + \frac{1-(-1)^d}{2})$ over $\Gamma_0(2)$, while $P_2(\tau)$ is a modular form of weight $d - (2n + \frac{1-(-1)^d}{2})$ over $\Gamma^0(2)$. Moreover, the following identity holds,

$$P_1(-\frac{1}{\tau}) = 2^d \tau^{d-(2n+\frac{1-(-1)^d}{2})} P_2(\tau). \tag{3.14}$$

Observe that at any point $x \in M$, up to the volume form determined by the metric on T_xM, both $P_i(\tau)$, $i = 1, 2$, can be view as a power series of $q^{\frac{1}{2}}$ with real Fourier coefficients. By Lemma 2.2, we have

$$P_2(\tau) = h_0(8\delta_2)^{m_1} + h_1(8\delta_2)^{m_1-2}\varepsilon_2 + \cdots + h_{[\frac{m_1}{2}]}(8\delta_2)^{m_1-2[\frac{m_1}{2}]}\varepsilon_2^{[\frac{m_1}{2}]}, \tag{3.15}$$

where each h_j, $0 \le j \le [\frac{m_1}{2}]$, is a real multiple of the volume form at x. By (2.19) (3.14) and (3.15), we get

$$P_1(\tau) = 2^d \left[h_0(8\delta_1)^{m_1} + h_1(8\delta_1)^{m_1-2}\varepsilon_1 + \cdots + h_{[\frac{m_1}{2}]}(8\delta_1)^{m_1-2[\frac{m_1}{2}]}\varepsilon_1^{[\frac{m_1}{2}]}\right]. \tag{3.16}$$

By comparing the constant term in (3.16), we get

$$\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2\frac{c}{2}} \frac{(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}}{(\cosh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}}\right\}^{(2d)} = 2^{\frac{3}{2}d-n-\frac{1-(-1)^d}{4}} \sum_{r=0}^{[\frac{m_1}{2}]} 2^{-6r} h_r. \tag{3.17}$$

By comparing the coefficients of $q^{\frac{j}{2}}$, $j \ge 0$ between the two sides of (3.15), we can use the induction method to prove that each h_r $0 \le r \le [\frac{m_1}{2}]$, can be expressed through a canonical integral linear combination of

$$\left\{\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}} \mathrm{ch}(B_r(T_CM,(2n+\frac{1-(-1)^d}{2})\xi_C^0,\xi_C))\right\}^{(2d)}.$$

Here we write out the explicit expressions for h_0 and h_1 as follows.

$$h_0 = (-1)^{m_1}\left\{\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}\right\}^{(2d)}, \tag{3.18}$$

$$h_1 = (-1)^{m_1}\left\{\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}\right.$$

$$\cdot\left(\mathrm{ch}(B_1(T_CM,(2n+\frac{1-(-1)^d}{2})\xi_C^0,\xi_C))-24m_1\right)\Bigg\}. \tag{3.19}$$

□

Putting $d=4k+2$ and $n=0$ in Theorem 3.1, we get the Han-Zhang cancellation formula (cf. [HZ2]),

Corollary 3.2 *The following cancellation formula holds*

$$\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2\frac{c}{2}}\right\}^{(8k+4)}=8\sum_{r=0}^{k}2^{6k-6r}\left\{\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}\mathrm{ch}(b_r(T_CM,\xi_C))\right\}^{(8k+4)}. \tag{3.20}$$

If ξ is a trivial bundle, we get the Han-Huang cancellation formula (cf. [HH]),

Corollary 3.3 *The following cancellation formula holds*

$$\left\{\widehat{L}(TM,\nabla^{TM})\frac{(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}}{(\cosh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}}\right\}^{(2d)}$$

$$=2^{\frac{3}{2}d-n-\frac{1-(-1)^d}{4}}\sum_{r=0}^{[\frac{m_1}{2}]}2^{-6r}\left\{\widehat{A}(TM,\nabla^{TM})(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}\right.$$

$$\left.\cdot\mathrm{ch}(b_r(T_CM,(2n+\frac{1-(-1)^d}{2})\xi_C^0,C^2))\right\}^{(2d)}, \tag{3.21}$$

Putting $d=6$ and $n=1$, i.e. for 12-dimensional manifold M, we have

Corollary 3.4 *The following cancellation formula holds*

$$\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2\frac{c}{2}}\frac{(\sinh\frac{c_0}{2})^2}{(\cosh\frac{c_0}{2})^2}\right\}^{(12)}=\left\{\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})^2\right.$$

$$\left.\cdot\Big(112-4\mathrm{ch}(T_CM,\nabla^{T_CM})+8(e^{c_0}+e^{-c_0}-2)+12(e^{c}+e^{-c}-2)\Big)\right\}^{(12)}. \tag{3.22}$$

Putting $d=6$ and $n=2$, i.e. for 12-dimensional manifold M, we have

Corollary 3.5 *The following cancellation formula holds*

$$\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2\frac{c}{2}}\frac{(\sinh\frac{c_0}{2})^4}{(\cosh\frac{c_0}{2})^4}\right\}^{(12)}=-128\left\{\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})^4\right\}^{(12)}. \tag{3.23}$$

Putting $d=5$ and $n=0$, i.e. for 10-dimensional manifold M, we have

Corollary 3.6 *The following cancellation formula holds*

$$\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2\frac{c}{2}}\frac{(\sinh\frac{c_0}{2})}{(\cosh\frac{c_0}{2})}\right\}^{(10)}=\left\{\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})\right.$$

$$\left.\cdot\Big(52-2\mathrm{ch}(T_CM,\nabla^{T_CM})+2(e^{c_0}+e^{-c_0}-2)+6(e^c+e^{-c}-2)\Big)\right\}^{(10)}. \tag{3.24}$$

Putting $d=5$ and $n=1$, i.e. for 10-dimensional manifold M, we have

Corollary 3.6 *The following cancellation formula holds*

$$\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2\frac{c}{2}}\frac{(\sinh\frac{c_0}{2})^3}{(\cosh\frac{c_0}{2})^3}\right\}^{(10)}=-64\left\{\widehat{A}(TM,\nabla^{TM})\cosh\frac{c}{2}(\sinh\frac{c_0}{2})^3\right\}^{(10)}. \tag{3.25}$$

Nextly we go on to prove some cancellation formulas. Define

$$\Theta_1(T_CM+\xi_C,m_0\xi_C^0,\xi_C)=\bigotimes_{n=1}^{\infty}S_{q^n}(\widetilde{T_CM}+\widetilde{\xi_C}-m_0\widetilde{\xi_C^0})$$

$$\otimes\bigotimes_{m=1}^{\infty}\wedge_{q^m}(\widetilde{T_CM}+\widetilde{\xi_C}-m_0\widetilde{\xi_C^0}-2\widetilde{\xi_C})\otimes\bigotimes_{r=1}^{\infty}\wedge_{q^{r-\frac{1}{2}}}(\widetilde{\xi_C})\otimes\bigotimes_{s=1}^{\infty}\wedge_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_C}),$$

$$\Theta_2(T_CM+\xi_C,m_0\xi_C^0,\xi_C)=\bigotimes_{n=1}^{\infty}S_{q^n}(\widetilde{T_CM}+\widetilde{\xi_C}-m_0\widetilde{\xi_C^0})$$

$$\otimes\bigotimes_{m=1}^{\infty}\wedge_{-q^{m-\frac{1}{2}}}(\widetilde{T_CM}+\widetilde{\xi_C}-m_0\widetilde{\xi_C^0}-2\widetilde{\xi_C})\otimes\bigotimes_{r=1}^{\infty}\wedge_{q^{r-\frac{1}{2}}}(\widetilde{\xi_C})\otimes\bigotimes_{s=1}^{\infty}\wedge_{q^s}(\widetilde{\xi_C}), \tag{3.26}$$

$\Theta_1(T_CM+\xi_C,m_0\xi_C^0,\xi_C)$ and $\Theta_2(T_CM+\xi_C,m_0\xi_C^0,\xi_C)$ admit formal Fourier expansion in $q^{\frac{1}{2}}$ as

$$\Theta_1(T_CM+\xi_C,m_0\xi_C^0,\xi_C)=A_0'(T_CM,m_0\xi_C^0,\xi_C)+A_1'(T_CM,m_0\xi_C^0,\xi_C)q^{\frac{1}{2}}+\cdots,$$

$$\Theta_2(T_CM+\xi_C,m_0\xi_C^0,\xi_C)=B_0'(T_CM,m_0\xi_C^0,\xi_C)+B_1'(T_CM,m_0\xi_C^0,\xi_C)q^{\frac{1}{2}}+\cdots, \tag{3.27}$$

Set

$$Q_1'(\tau)=\widehat{L}(TM,\nabla^{TM})\frac{\cosh\frac{c}{2}}{\sinh\frac{c}{2}}\frac{(\sinh\frac{c_0}{2})^{2n+\frac{1+(-1)^d}{2}}}{(\cosh\frac{c_0}{2})^{2n+\frac{1+(-1)^d}{2}}}$$

$$\cdot\left(\mathrm{ch}(\Theta_1(T_CM+\xi_C,(2n+\frac{1+(-1)^d}{2})\xi_C^0,C^2))\right.$$

$$\left.-\frac{\mathrm{ch}(\Theta_1(T_CM+\xi_C,(2n+\frac{1+(-1)^d}{2})\xi_C^0,\xi_C)}{\cosh^2\frac{c}{2}}\right), \tag{3.28}$$

$$Q_2'(\tau)=\widehat{A}(TM,\nabla^{TM})\frac{1}{2\sinh\frac{c}{2}}(\sinh\frac{c_0}{2})^{2n+\frac{1+(-1)^d}{2}}$$

$$\cdot\Bigg(\mathrm{ch}(\Theta_2(T_CM+\xi_C,(2n+\frac{1+(-1)^d}{2})\xi_C^0,C^2))$$

$$-\cosh(\frac{c}{2})\mathrm{ch}(\Theta_2(T_CM+\xi_C,(2n+\frac{1+(-1)^d}{2})\xi_C^0,\xi_C))\Bigg)\,. \tag{3.29}$$

Direct computations show that

$$Q_1'(\tau)=2^d(\pi\sqrt{-1})^{2n+\frac{1+(-1)^d}{2}-1}\left(\prod_{j=1}^d x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\frac{\theta_1(x_j,\tau)}{\theta_1(0,\tau)}\right)\left(\frac{\theta(u',\tau)}{\theta'(0,\tau)}\frac{\theta_1(0,\tau)}{\theta_1(u',\tau)}\right)^{2n+\frac{1+(-1)^d}{2}}$$

$$\cdot\frac{\theta'(0,\tau)}{\theta(u,\tau)}\left(\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}-\frac{\theta_1(0,\tau)}{\theta_1(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\right), \tag{3.30}$$

$$Q_2'(\tau)=\frac{1}{2}(\sqrt{-1}\pi)^{2n+\frac{1+(-1)^d}{2}-1}\left(\prod_{j=1}^d x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\frac{\theta_2(x_j,\tau)}{\theta_2(0,\tau)}\right)\left(\frac{\theta(u',\tau)}{\theta'(0,\tau)}\frac{\theta_2(0,\tau)}{\theta_2(u',\tau)}\right)^{2n+\frac{1+(-1)^d}{2}}$$

$$\cdot\frac{\theta'(0,\tau)}{\theta(u,\tau)}\left(\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}-\frac{\theta_2(0,\tau)}{\theta_2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\right). \tag{3.31}$$

Let $P_1'(\tau)=Q_1'(\tau)^{(2d)},\ P_2'(\tau)=Q_2'(\tau)^{(2d)}$, similarly we have $P_1'(\tau)$ is a modular form of weight $d+1-(2n+\frac{1+(-1)^d}{2})$ over $\Gamma_0(2)$, while $P_2'(\tau)$ is a modular form of weight $d+1-(2n+\frac{1+(-1)^d}{2})$ over $\Gamma^0(2)$. Moreover, the following identity holds,

$$P_1'(-\frac{1}{\tau})=2^{d+1}\tau^{d+1-(2n+\frac{1+(-1)^d}{2})}P_2'(\tau). \tag{3.32}$$

Let n is a nonnegative integer and satisfy $d-1-(2n+\frac{1+(-1)^d}{2})>0$. Using the same trick in the proof of Theorem 3.1, we obtain

Theorem 3.7 *The following identity holds,*

$$\left\{\widehat{L}(TM,\nabla^{TM})\frac{\sinh\frac{c}{2}}{\cosh\frac{c}{2}}\frac{(\sinh\frac{c_0}{2})^{2n+\frac{1+(-1)^d}{2}}}{(\cosh\frac{c_0}{2})^{2n+\frac{1+(-1)^d}{2}}}\right\}^{(2d)}=2^{\frac{3}{2}(d+1)-n-\frac{1+(-1)^d}{4}}\sum_{r=0}^{[\frac{m_2}{2}]}2^{-6r}h_r, \tag{3.33}$$

where $m_2=\frac{d+1}{2}-n-\frac{1+(-1)^d}{4}$ *and each* h_r, $0\le r\le[\frac{m_2}{2}]$, *is a canonical integral linear combination of*

$$\Bigg\{\widehat{A}(TM,\nabla^{TM})\frac{1}{2\sinh\frac{c}{2}}(\sinh\frac{c_0}{2})^{2n+\frac{1+(-1)^d}{2}}\Bigg(\mathrm{ch}(B_j'(T_CM,(2n+\frac{1+(-1)^d}{2})\xi_C^0,C^2))$$

$$-\cosh(\frac{c}{2})\mathrm{ch}(B_j'(T_CM,(2n+\frac{1+(-1)^d}{2})\xi_C^0,\xi_C))\Bigg)\Bigg\}^{(2d)},\ 0\le j\le r.$$

Putting $d=4k+1$ and $n=0$ in Theorem 3.7, we get the Han-Huang cancellation formula (cf. [HH]),

Corollary 3.8 *The following cancellation formula holds*

$$\left\{\widehat{L}(TM,\nabla^{TM})\frac{\sinh\frac{c}{2}}{\cosh\frac{c}{2}}\right\}^{(8k+2)}=8\sum_{r=0}^{k}2^{6k-6r}h_r. \tag{3.34}$$

Putting $d=6$ and $n=1$, i.e. for 12-dimensional manifold M, we have

Corollary 3.9 *The following cancellation formula holds*

$$\begin{aligned}\left\{\widehat{L}(TM,\nabla^{TM})\frac{\sinh\frac{c}{2}}{\cosh\frac{c}{2}}\frac{(\sinh\frac{c_0}{2})^3}{(\cosh\frac{c_0}{2})^3}\right\}^{(12)} &= \left\{\widehat{A}(TM,\nabla^{TM})\frac{1}{2\sinh\frac{c}{2}}(\sinh\frac{c_0}{2})^3\right.\\ &\cdot\left[\left(224+24(e^{c_0}+e^{-c_0}-2)-8\mathrm{ch}(T_CM,\nabla^{T_CM})\right)(1-\cosh\frac{c}{2})\right.\\ &\left.\left.-8(e^c+e^{-c}-2)(1+2\cosh\frac{c}{2})\right]\right\}^{(12)}.\end{aligned} \tag{3.35}$$

Putting $d=5$ and $n=1$, i.e. for 10-dimensional manifold M, we have

Corollary 3.10 *The following cancellation formula holds*

$$\begin{aligned}\left\{\widehat{L}(TM,\nabla^{TM})\frac{\sinh\frac{c}{2}}{\cosh\frac{c}{2}}\frac{(\sinh\frac{c_0}{2})^2}{(\cosh\frac{c_0}{2})^2}\right\}^{(10)} &= \left\{\widehat{A}(TM,\nabla^{TM})\frac{1}{2\sinh\frac{c}{2}}(\sinh\frac{c_0}{2})^2\right.\\ &\cdot\left[\left(104+8(e^{c_0}+e^{-c_0}-2)-4\mathrm{ch}(T_CM,\nabla^{T_CM})\right)(1-\cosh\frac{c}{2})\right.\\ &\left.\left.-(e^c+e^{-c}-2)(1+2\cosh\frac{c}{2})\right]\right\}^{(10)}.\end{aligned} \tag{3.36}$$

4 Transgressed forms and modularities

In this section, following [CH], we transgress the modular characteristic forms in Section 3 and then get some cancellation formulas.

Let M be $(2d-1)$-dimensional manifold. Set

$$\Theta_1(T_CM,m_0\xi_C^0)=\Theta_1(T_CM,m_0\xi_C^0,C^2);\ \Theta_2(T_CM,m_0\xi_C^0)=\Theta_2(T_CM,m_0\xi_C^0,C^2);$$

$$\Theta_3(T_CM, m_0\xi^0_C) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_CM} - m_0\widetilde{\xi^0_C}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^{m-\frac{1}{2}}}(\widetilde{T_CM} - m_0\widetilde{\xi^0_C}).$$

Set

$$\Phi_L(\tau) = \widehat{L}(TM, \nabla^{TM}) \frac{(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}}{(\cosh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}}} \mathrm{ch}(\Theta_1(T_CM, (2n + \frac{1-(-1)^d}{2})\xi^0_C)), \quad (4.1)$$

$$\Phi_W(\tau) = \widehat{A}(TM, \nabla^{TM})(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}} \mathrm{ch}(\Theta_2(T_CM, (2n + \frac{1-(-1)^d}{2})\xi^0_C)), \quad (4.2)$$

$$\Phi'_W(\tau) = \widehat{A}(TM, \nabla^{TM})(\sinh\frac{c_0}{2})^{2n+\frac{1-(-1)^d}{2}} \mathrm{ch}(\Theta_3(T_CM, (2n + \frac{1-(-1)^d}{2})\xi^0_C)). \quad (4.3)$$

Direct computations as in Section 3 and applying Chern-weil theory, we have

$$\Phi_L(\nabla^{TM}, \nabla^{\xi^0}, \tau) = \sqrt{2}^{2d-1}(\pi\sqrt{-1})^{2n+\frac{1-(-1)^d}{2}} \det{}^{\frac{1}{2}} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0,\tau)}{\theta(\frac{R^{TM}}{4\pi^2},\tau)} \frac{\theta_1(\frac{R^{TM}}{4\pi^2},\tau)}{\theta_1(0,\tau)} \right)$$

$$\cdot \left(\frac{\theta(u',\tau)}{\theta'(0,\tau)} \frac{\theta_1(0,\tau)}{\theta_1(u',\tau)} \right)^{2n+\frac{1-(-1)^d}{2}}, \quad (4.4)$$

$$\Phi_W(\nabla^{TM}, \nabla^{\xi^0}, \tau) = (\pi\sqrt{-1})^{2n+\frac{1-(-1)^d}{2}} \det{}^{\frac{1}{2}} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0,\tau)}{\theta(\frac{R^{TM}}{4\pi^2},\tau)} \frac{\theta_2(\frac{R^{TM}}{4\pi^2},\tau)}{\theta_2(0,\tau)} \right)$$

$$\cdot \left(\frac{\theta(u',\tau)}{\theta'(0,\tau)} \frac{\theta_2(0,\tau)}{\theta_2(u',\tau)} \right)^{2n+\frac{1-(-1)^d}{2}}, \quad (4.5)$$

$$\Phi'_W(\nabla^{TM}, \nabla^{\xi^0}, \tau) = (\pi\sqrt{-1})^{2n+\frac{1-(-1)^d}{2}} \det{}^{\frac{1}{2}} \left(\frac{R^{TM}}{4\pi^2} \frac{\theta'(0,\tau)}{\theta(\frac{R^{TM}}{4\pi^2},\tau)} \frac{\theta_3(\frac{R^{TM}}{4\pi^2},\tau)}{\theta_3(0,\tau)} \right)$$

$$\cdot \left(\frac{\theta(u',\tau)}{\theta'(0,\tau)} \frac{\theta_3(0,\tau)}{\theta_3(u',\tau)} \right)^{2n+\frac{1-(-1)^d}{2}}, \quad (4.6)$$

where $u' = \frac{\sqrt{-1}\mathrm{Pf}(R^{\xi^0})}{2\pi}$ (cf. [Z]). As in [CH] and [W], we transgress Φ_L, Φ_W, Φ'_W and get the following forms:
$CS\Phi_L(\nabla^{TM}_0, \nabla^{TM}_1, \nabla^{\xi^0}, \tau)$

$$:= \frac{\sqrt{2}}{8\pi^2}\int_0^1 \Phi_L(\nabla^{TM}_t, \nabla^{\xi^0}, \tau)\mathrm{tr}\left[A\left(\frac{1}{\frac{R^{TM}_t}{4\pi^2}} - \frac{\theta'(\frac{R^{TM}_t}{4\pi^2},\tau)}{\theta(\frac{R^{TM}_t}{4\pi^2},\tau)} + \frac{\theta'_1(\frac{R^{TM}_t}{4\pi^2},\tau)}{\theta_1(\frac{R^{TM}_t}{4\pi^2},\tau)} \right) \right] dt; \quad (4.7)$$

$CS\Phi_W(\nabla^{TM}_0, \nabla^{TM}_1, \nabla^{\xi^0}, \tau)$

$$:= \frac{1}{8\pi^2}\int_0^1 \Phi_W(\nabla^{TM}_t, \nabla^{\xi^0}, \tau)\mathrm{tr}\left[A\left(\frac{1}{\frac{R^{TM}_t}{4\pi^2}} - \frac{\theta'(\frac{R^{TM}_t}{4\pi^2},\tau)}{\theta(\frac{R^{TM}_t}{4\pi^2},\tau)} + \frac{\theta'_2(\frac{R^{TM}_t}{4\pi^2},\tau)}{\theta_2(\frac{R^{TM}_t}{4\pi^2},\tau)} \right) \right] dt; \quad (4.8)$$

$$CS\Phi'_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)$$

$$:=\frac{1}{8\pi^2}\int_0^1\Phi'_W(\nabla_t^{TM},\nabla^{\xi^0},\tau)\mathrm{tr}\left[A\left(\frac{1}{\frac{R_t^{TM}}{4\pi^2}}-\frac{\theta'(\frac{R_t^{TM}}{4\pi^2},\tau)}{\theta(\frac{R_t^{TM}}{4\pi^2},\tau)}+\frac{\theta_3'(\frac{R_t^{TM}}{4\pi^2},\tau)}{\theta_3(\frac{R_t^{TM}}{4\pi^2},\tau)}\right)\right]dt, \quad (4.9)$$

which lie in $\Omega^{\mathrm{odd}}(M,\mathbf{C})[[q^{\frac{1}{2}}]]$ and their top components represent elements in $H^{2d-1}(M,\mathbf{C})[[q^{\frac{1}{2}}]]$. We have the following results.

Theorem 4.1 *Let M be a $2d-1$ dimensional manifold and ∇_0^{TM}, ∇_1^{TM} be two connections on TM and ξ^0 be a two dimensional oriented Euclidean real vector bundle with a Euclidean connection ∇^{ξ^0}, then we have*
1) $\{CS\Phi_L(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)\}^{(2d-1)}$ is a modular form of weight $d-(2n+\frac{1-(-1)^d}{2})$ over $\Gamma_0(2)$;
$\{CS\Phi_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)\}^{(2d-1)}$ is a modular form of weight $d-(2n+\frac{1-(-1)^d}{2})$ over $\Gamma^0(2)$;
$\{CS\Phi'_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)\}^{(2d-1)}$ is a modular form of weight $d-(2n+\frac{1-(-1)^d}{2})$ over $\Gamma_\theta(2)$.
2) The following equalities hold,

$$\{CS\Phi_L(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},-\frac{1}{\tau})\}^{(2d-1)}=2^d\tau^{d-(2n+\frac{1-(-1)^d}{2})}\{CS\Phi_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)\}^{(2d-1)},$$

$$CS\Phi_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau+1)=CS\Phi'_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau).$$

Proof. By (2.12)-(2.17) and (4.7)-(4.9), we have

$$\{CS\Phi_L(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},-\frac{1}{\tau})\}^{(2d-1)}=2^d\tau^{d-(2n+\frac{1-(-1)^d}{2})}\{CS\Phi_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)\}^{(2d-1)},$$

$$CS\Phi_L(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau+1)=CS\Phi_L(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau),$$

$$\{CS\Phi_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},-\frac{1}{\tau})\}^{(2d-1)}=2^{-d}\tau^{d-(2n+\frac{1-(-1)^d}{2})}\{CS\Phi_L(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)\}^{(2d-1)},$$

$$CS\Phi_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau+1)=CS\Phi'_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau),$$

$$\{CS\Phi'_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi},-\frac{1}{\tau})\}^{(2d-1)}=(\tau)^{d-(2n+\frac{1-(-1)^d}{2})}\{CS\Phi'_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)\}^{(2d-1)},$$

$$CS\Phi'_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau+1)=CS\Phi_W(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau). \quad (4.10)$$

From (4.10), we can prove Theorem 4.1. □

Let $d=4$ and $n=1$, i.e. for 7-dimensional manifold, $\{CS\Phi_L(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)\}^7$ is a modular form of weight 2 over $\Gamma_0(2)$. Set $A=\nabla_1^{TM}-\nabla_0^{TM}$. Using similar discussions in [CH, p.15], we get

$$CS\Phi_L(\nabla_0^{TM},\nabla_1^{TM},\nabla^{\xi^0},\tau)=-\frac{\delta_1}{6\pi^2}c_0^2\mathrm{tr}\left[A[\nabla_0^{TM},\nabla_1^{TM}]+\frac{2}{3}A\wedge A\wedge A\right]. \quad (4.11)$$

Similarly, we obtain that

$$CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^{\xi^0}, \tau) = -\frac{\delta_2}{96\pi^2} c_0^2 \mathrm{tr}\left[A[\nabla_0^{TM}, \nabla_1^{TM}] + \frac{2}{3} A \wedge A \wedge A\right]. \quad (4.12)$$

$$CS\Phi'_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^{\xi^0}, \tau) = -\frac{\delta_3}{96\pi^2} c_0^2 \mathrm{tr}\left[A[\nabla_0^{TM}, \nabla_1^{TM}] + \frac{2}{3} A \wedge A \wedge A\right]. \quad (4.12)$$

From Theorem 4.1, we can imply some twisted cancellation formulas for odd dimensional manifolds. For example, let $d = 6$ and $n = 1$, i.e. M be 11 dimensional. We have that $\{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^{\xi^0}, \tau)\}^{(11)}$ is a modular form of weight 4 over $\Gamma_0(2)$, $\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^{\xi^0}, \tau)\}^{(11)}$ is a modular form of weight 4 over $\Gamma^0(2)$ and

$$\{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^{\xi^0}, -\frac{1}{\tau})\}^{(11)} = 2^6\tau^4\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^{\xi^0}, \tau)\}^{(11)}. \quad (4.13)$$

By Lemma 2.2, we have

$$\{CS\Phi_W(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^{\xi^0}, \tau)\}^{(11)} = z_0(8\delta_2)^2 + z_1\varepsilon_2, \quad (4.14))$$

and by (2.19) and Theorem (4.13),

$$\{CS\Phi_L(\nabla_0^{TM}, \nabla_1^{TM}, \nabla^{\xi}, \tau)\}^{(11)} = 2^6[z_0(8\delta_1)^2 + z_1\varepsilon_1]. \quad (4.15)$$

By comparing the $q^{\frac{1}{2}}$-expansion coefficients in (4.14), we get

$$z_0 = \left\{\int_0^1 \widehat{A}(TM, \nabla_t^{TM})(\sinh\frac{c_0}{2})^2 \mathrm{tr}\left[A\left(\frac{1}{2R_t^{TM}} - \frac{1}{8\pi\tan\frac{R_t^{TM}}{4\pi}}\right)\right] dt\right\}^{(11)}, \quad (4.16)$$

$$\begin{aligned} z_1 = \Bigg\{\int_0^1 \widehat{A}(TM, \nabla_t^{TM})(\sinh\frac{c_0}{2})^2 &\left(-\mathrm{ch}(T_CM, \nabla_t^{T_CM}) + 2(e^{c_0} + e^{-c_0}) - 41\right) \\ &\times \mathrm{tr}\left[A\left(\frac{1}{2R_t^{TM}} - \frac{1}{8\pi\tan\frac{R_t^{TM}}{4\pi}}\right)\right] dt + \\ &\int_0^1 \widehat{A}(TM, \nabla_t^{TM})(\sinh\frac{c_0}{2})^2 \mathrm{tr}\left[\frac{A}{2\pi}\sin\frac{R_t^{TM}}{4\pi}\right] dt\Bigg\}^{(11)}. \end{aligned} \quad (4.17)$$

Plugging (4.16) and (4.17) into (4.15) and comparing the constant terms of both sides, we obtain that

$$\left\{\int_0^1 \sqrt{2}\widehat{L}(TM, \nabla_t^{TM})\frac{\sinh^2\frac{c_0}{2}}{\cosh^2\frac{c_0}{2}} \mathrm{tr}\left[A\left(\frac{1}{2R_t^{TM}} - \frac{1}{4\pi\sin\frac{R_t^{TM}}{2\pi}}\right)\right]\right\}^{(11)} = 2^2(2^6 z_0 + z_1), \quad (4.18)$$

so we have the following 11-dimensional analogue of the twisted miraculous cancellation formula.

Corollary 3.3 *The following equality holds*

$$\left\{\int_0^1 \sqrt{2}\widehat{L}(TM,\nabla_t^{TM})\frac{\sinh^2\frac{c_0}{2}}{\cosh^2\frac{c_0}{2}}\mathrm{tr}\left[A\left(\frac{1}{2R_t^{TM}}-\frac{1}{4\pi\sin\frac{R_t^{TM}}{2\pi}}\right)\right]\right\}^{(11)}$$

$$= 4\left\{\int_0^1 \widehat{A}(TM,\nabla_t^{TM})(\sinh\frac{c_0}{2})^2\left(-\mathrm{ch}(T_CM,\nabla_t^{T_CM})+2(e^{c_0}+e^{-c_0})+23\right)\right.$$

$$\times\mathrm{tr}\left[A\left(\frac{1}{2R_t^{TM}}-\frac{1}{8\pi\tan\frac{R_t^{TM}}{4\pi}}\right)\right]dt+$$

$$\left.\int_0^1 \widehat{A}(TM,\nabla_t^{TM})(\sinh\frac{c_0}{2})^2\mathrm{tr}\left[\frac{A}{2\pi}\sin\frac{R_t^{TM}}{4\pi}\right]dt\right\}^{(11)}. \qquad (4.19)$$

let $d=5$ and $n=0$, i.e. M be 9 dimensional. Using similar discussions, we have

Corollary 3.4 *The following equality holds*

$$\left\{\int_0^1 \sqrt{2}\widehat{L}(TM,\nabla_t^{TM})\frac{\sinh\frac{c_0}{2}}{\cosh\frac{c_0}{2}}\mathrm{tr}\left[A\left(\frac{1}{2R_t^{TM}}-\frac{1}{4\pi\sin\frac{R_t^{TM}}{2\pi}}\right)\right]\right\}^{(9)}$$

$$= 2\left\{\int_0^1 \widehat{A}(TM,\nabla_t^{TM})(\sinh\frac{c_0}{2})\left(-\mathrm{ch}(T_CM,\nabla_t^{T_CM})+(e^{c_0}+e^{-c_0})+23\right)\right.$$

$$\times\mathrm{tr}\left[A\left(\frac{1}{2R_t^{TM}}-\frac{1}{8\pi\tan\frac{R_t^{TM}}{4\pi}}\right)\right]dt+$$

$$\left.\int_0^1 \widehat{A}(TM,\nabla_t^{TM})(\sinh\frac{c_0}{2})\mathrm{tr}\left[\frac{A}{2\pi}\sin\frac{R_t^{TM}}{4\pi}\right]dt\right\}^{(9)}. \qquad (4.20)$$

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